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A New Fixed-Point Theorem in b-Metric Space with Application

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Abstract

The goal of this study is to provide a new contraction mapping in b-metric space. We build a fixed point result for this new contraction mapping; we also provide applications related to authenticity concerns in integral equations.

Keywords: *Fixed Point, $\theta-$Contraction, B−Metric Space, $\theta-\phi-$Contraction, Complete Metric Space.*

Subject Classification: *47H10, 54H25.*

1. Introduction

One of the most important theorems and helpful tools in the investigation of metric spaces is Banach's contraction principle. Different fixed-point theorems have been established in these spaces and several generalizations of the idea of metric space have been built, (see [4], [8], [12], [13]). In particular, the b-metric spaces were introduced by Bakhtin [2] and Czerwik [6] so that the triangle inequality was replaced by the b-triangle inequality. A b-metric space is not always a metric space, but any metric space is also a b-metric space. Numerous fixed-point results have been found on these spaces (see [7], [14], [15]). In a novel class of contraction mappings called $\theta-$contraction (or JS-contraction) introduced by Jleli and Samet [7] in 2014, they showed a fixed point result in generalized metric spaces. In this direction, several fixed point theorems have been created, investigated, and generalised in a variety of conditions (see [9]–[11]). Most recently, Rossafi et al. [5] developed a new concept of "$\theta-\phi-$contraction" and some fixed point discovering for these mappings in metric space and generalized the conclusions obtained by Zheng et al [16]. In this paper, we demonstrate the fixed-point theorem in b-metric space and introduce a novel concept known as generalized $\theta-\phi-$contraction. The results of this study develop the corresponding findings in b−metric space from Kannan [8], Reich [13], and Rossafi [15]. As an application, we discuss the existence and uniqueness of a solution to the nonlinear Fredholm integral equations [1-3].

2. Preliminaries

In this section, we recall some definitions and Lemmas associated with our work.

Definition 2.1. [6]. Let S be a non-empty set. A mapping $d_b : S\times S \to [0,+\infty)$ is said to be a b-metric, if there exists b ≥ 1 such that db satisfies the following conditions:

1. $d_b(s,t)=0,$ if and only if $s=t,$
2. $d_b(s,t)=d_b t, s),$
3. $d_b(s,t)\le b[d_b(s,r)+d_b(r,t)],$

For all $s,t,r\in S$. The pair is called a *b*-metric space.

The following Lemmas will be used to establish our result.

Lemma 2.2. [1] Let (*S*, *d*) be *b*- metric space and

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suppose that $r_n \to r$ and $s_n \to s$ as $n \to \infty$ with $r \neq s, r_n \neq r$ and $s_n \neq s, \forall n,$ then we have

$$\frac{1}{b^2} d(r,s) \leq \lim_{n\to\infty} \inf\, d(r_n, s_n) \leq \lim_{n\to\infty} \sup d(r_n, s_n) \leq b^2 d(r,s).$$

In particular, if $r = s,$ then we have $\lim_{n\to\infty} d(r_n, s_n) = 0.$ Moreover for each $t \in S$ we have

$$\frac{1}{b} d(r,t) \leq \lim_{n\to\infty} \inf\, d(r_n, t) \leq \lim_{n\to\infty} \sup d(r_n, t) \leq bd(r,t)$$

For all $r \in S.$

Lemma 2.3. [14] Let (S,d) be b-metric space and let $\{r_n\}$ be a sequence in S such that $\lim_{n\to\infty} d(r_n, r_{n+1}) = 0.$

If $\{r_n\}$ is not a b-Cauchy sequence, then there exist $\in > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that

$$\in \leq \lim_{k\to\infty} \inf\, d(r_{m(k)}, r_{n(k)}) \leq \lim_{k\to\infty} \sup d(r_{m(k)}, r_{n(k)}) \leq b \in,$$

$$\in \leq \lim_{k\to\infty} \inf\, d(r_{n(k)}, r_{m(k)+1}) \leq \lim_{k\to\infty} \sup d(r_{n(k)}, r_{m(k)+1}) \leq b \in,$$

$$\in \leq \lim_{k\to\infty} \inf\, d(r_{m(k)}, r_{n(k)+1}) \leq \lim_{k\to\infty} \sup d(r_{m(k)}, r_{n(k)+1}) \leq b \in,$$

$$\frac{\in}{b} \leq \lim_{k\to\infty} \inf\, d(r_{m(k)+1}, r_{n(k)+1}) \leq \lim_{k\to\infty} \sup d(r_{m(k)+1}, r_{n(k)+1}) \leq b^2 \in.$$

Definition 2.4. [7] Let Θ be the family of all functions $\theta : (0,\infty) \to (1,\infty)$ such that $(\theta_1)\ \theta$ is increasing, (θ_2) for each sequence $\{r_n\} \subset (0,\infty);$ $\lim_{n\to\infty} r_n = 0$ if and only if $\lim_{n\to\infty} \theta(r_n) = 1;$ $(\theta_3)\ \theta$ is continuous.

Definition 2.5. [7] Let Φ be the family of all functions $\phi : [1,\infty) \to [1,\infty)$ such that $(\phi_1)\ \phi$ is nondecreasing, (ϕ_2) for each $t \in (1,\infty);$ $\lim_{n\to\infty} \phi^n(t) = 1;$ $(\phi_3)\ \phi$ is continuous.

Lemma 2.6. [7] If $\phi \in \Phi,$ then $\phi(1) = 1$ and $\phi(t) < t \forall t \in (1,\infty).$

Definition 2.7. [15] Let (S,d) be a b-metric space with parameter $b > 1$ and $T : S \to S$ be a mapping then

1. T is said to be a $\theta-$contraction if there exist $\theta \in \Theta$ and $t \in (0,1)$ and $a \in [0,1]$ such that

$$d(Tr, Ts) > 0 \Rightarrow \theta[b^3 d(Tr, Ts)] \leq \theta[M(r,s)]^t,$$

where

$$M(r,s) = \max\{d(r,s), d(r,Tr), d(s,Ts), \frac{d(r,Ts) + d(Tr,s)}{2b^2}\}.$$

2. T is said to be a $\theta-\phi-$contraction if there exist $\theta \in \Theta$ and $\phi \in \Phi$ such that

$$d(Tr, Ts) > 0 \Rightarrow \theta[b^3 d(Tr, Ts)] \leq \phi[\theta(M(r,s))],$$

where

$$M(r,s) = a\max\{d(r,s), d(r,Tr), d(s,Ts), \frac{d(r,Ts) + d(Tr,s)}{2b^2}\}.$$

3. Main Results

In this section we introduce Gregus type contraction in b-metric space and establish fixed point result for such contraction. Now, we present our result for an Gregus type contraction in a complete b-metric space endowed with an example:

Definition 3.1. Let (S,d) be a b-metric space with parameter $b > 1$ and $T : S \to S$ be a mapping. T is said to be a Gregus type $\theta-\phi-$contraction if there exist $\theta \in \Theta$ and $\phi \in \Phi$ and $a \in [0,1]$ such that

$$d(Tr, Ts) > 0 \Rightarrow \theta[b^3 d(Tr, Ts)] \leq \phi[\theta(M(r,s))], \tag{3.1}$$

where

$$M(r,s) = a\max\{d(r,s), d(r,Tr), d(s,Ts), \frac{d(r,Ts) + d(Tr,s)}{2b^2}\} + (1-a)d(r,s).$$

Example 1. Let $S=[1,\infty)$. Define $d:S\times S\to[0,\infty)$ by $d(r,s)=|r-s|^2$. Then (S,d) is a b-metric space with coefficient $b=2$. Define a mapping $T:S\to S$ by $T(r)=r^{\frac{1}{8}}$. Evidently, $T(r)\in S$. Let $\theta(r)=e^{\sqrt{r}}$, $\phi(r)=e^{\frac{r+1}{2}}$. It is obvious that $\theta\in\Theta$ and $\phi\in\Phi$.

Theorem 3.2. Let (S,d) be a complete b-metric space with $b\ge 1$ and $T:S\to S$ be a Gregus type $\theta-\phi-$contraction then T has a unique fixed point.

Proof. Take an arbitrary point $r_0\in S$ and define a sequence $\{r_n\}$ by

$$r_{n+1}=Tr_n=T^{n+1}r_0,$$

for all $n\in N$. If there exist $n_0\in N$ such that $d(r_{n_0},r_{n_{-+1}})=0$ then the proof is completed. Suppose that $d(r_{n_0},r_{n_{0+1}})>0$ for all $n\in N$. Letting $r=r_{n-1}$ and $s=r_n$ in (3.1), we have

$$\theta[d(r_n,r_{n+1})]\le\theta[b^3d(r_n,r_{n+1})]\le\phi[\theta(M(r_{n-1},r_n))],\quad \forall n\in N, \tag{3.2}$$

Where

$$M(r_{n-1},r_n)=a\max\{d(r_{n-1},r_n),d(r_{n-1},r_n),d(r_n,r_{n+1}),\frac{d(r_n,r_n)+d(r_{n-1},r_{n+1})}{2b^3}\}+(1-a)\,d(r_{n-1},r_n)$$

$$=a\max\left\{d(r_{n-1},r_n),d(r_n,r_{n+1}),\frac{d(r_{n-1},r_{n+1}}{2b^2}\right\}+(1-a)\,d(r_{n-1},r_n).$$

Since

$$\frac{1}{2b^2}d(r_{n-1},r_{n+1})\le\frac{1}{2b^2}[b(d(r_{n-1},r_n)+d(r_n,r_{n+1}))]$$
$$=\frac{1}{2b}(d(r_{n-1},r_n)+d(r_n,r_{n+1}))$$
$$\le\frac{1}{2}(d(r_{n-1},r_n)+d(r_n,r_{n+1}))$$
$$\le\max\{d(r_{n-1},r_n),d(r_n,r_{n+1})\},$$

We obtain

$$M(r_{n-1},r_n)=a\max\{d(r_{n-1},r_n),d(r_n,r_{n+1})\}+(1-a)d(r_{n-1},r_n).$$

If $M(r_{n-1},r_n)=a\,d(r_n,r_{n+1})+(1-a)d(r_{n-1},r_n)$, then by (3.2), we have

$$\theta(d(r_n,r_{n+1}))\le\phi[\theta\{a(d(r_n,r_{n+1}))+(1-a)d(r_{n-1},r_n)\}]<\theta(ad(r_n,r_{n+1})),$$

which is a contradiction. Hence $M(r_{n-1},r_n)=d(r_{n-1},r_n)$. Thus

$$\theta(d(r_n,r_{n+1}))\le\phi[\theta(d(r_{n-1},r_n))]. \tag{3.3}$$

Repeating this step, we conclude that

$$\theta(d(r_n,r_{n+1}))\le\phi[\theta(d(r_{n-1},r_n))]\le\phi^2[\theta(d(r_{n-2},r_{n-1}))]\le\ldots\le\phi^n[\theta(d(r_0,r_1))].$$

From (3.3) and using (θ_1) we get

$$d(r_n,r_{n+1})<d(r_{n-1},r_n).$$

Therefore, $d(r_n,r_{n+1})_{n\in N}$ is a monotone strictly decreasing sequence of nonnegative real numbers. Consequently, there exists $\alpha\ge 0$ such that

$$\lim_{n\to\infty}d(r_{n+1},r_n)=\alpha.$$

Now, we claim that $\alpha=0$. Assume that $\alpha>0$. Since $d(r_n,r_{n+1})_{n\in N}$ is a nonnegative decreasing sequence, we have

$$d(r_n,r_{n+1})\ge\alpha,\forall n\in N.$$

By the property of θ, we get

$$1<\theta(\alpha)\le\theta(d(r_{n-1},r_n))\le\phi^n\theta(d(r_0,r_1))\le\ldots\le\phi^n\theta(d(r_0,r_1)). \tag{3.4}$$

Letting $n\to\infty$, we obtain

$$1<\theta(\alpha)\le\lim_{n\to\infty}\phi^n\theta(d(r_0,r_1))=1.$$

This is contradiction. Therefore,

$$\lim_{n\to\infty}d(r_n,r_{n+1})=0. \tag{3.5}$$

Next, we shall prove that $\{r_n\}_{n\in N}$ is a Cauchy sequence, i.e., $\lim_{n,m\to\infty}d(r_n,r_m)=0$. By Lemma 2.3, there is an $\in>0$ such that for an integer k there exist two sequences $\{n_{(k)}\}$ and $\{m_{(k)}\}$ such that

1. $\in\le\lim_{k\to\infty}\inf\,d(r_{m(k)},r_{n(k)})\le\lim_{k\to\infty}d(r_{m(k)},r_{n(k)})\le b\in$
2. $\frac{\in}{b}\le\lim_{k\to\infty}\inf\,d(r_{m(k)},r_{n(k)+1})$

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$$\leq \limsup_{k\to\infty} d(r_{m(k)}, r_{n(k)+1}) \leq b^2 \in$$

3. $$\frac{\in}{b} \leq \liminf_{k\to\infty} d(r_{m(k)+1}, r_{n(k)})$$

$$\leq \limsup_{k\to\infty} d(r_{m(k)+1}, r_{n(k)}) \leq b^2 \in$$

4. $$\frac{\in}{b^2} \leq \liminf_{k\to\infty} d(r_{m(k)+1}, r_{n(k)+1})$$

$$\leq \limsup_{k\to\infty} d(r_{m(k)+1}, r_{n(k)+1}) \leq b^3 \in.$$

From (3.1) and by setting $r = r_{m(k)}$ and $s = r_{n(k)}$, we have

$$M(r_{m(k)}, r_{n(k)}) = \max\{d(r_{m(k)}, r_{n(k)}), d(r_{m(k)}, r_{m(k)+1}), d(r_{n(k)}, r_{n(k)+1}), \frac{1}{2b^2}(d(r_{n(k)}, r_{m(k)+1}) + d(r_{m(k)}, r_{n(k)+1}))\}.$$

Taking the limit as $k \to \infty$ and using (3.5) and Lemma 2.3, we have

$$\lim_{k\to\infty} M(r_{m(k)}, r_{n(k)}) = \lim_{k\to\infty} \max\{d(r_{m(k)}, r_{n(k)}), d(r_{m(k)}, r_{m(k)+1}), d(r_{n(k)}, r_{n(k)+1}), \frac{1}{2b^2}(d(r_{n(k)}, r_{m(k)+1}) + d(r_{m(k)}, r_{n(k)+1}))\}$$

$$\leq \max\left\{b\in, 0, 0, \frac{1}{2b^2}(b^2\in + b^2\in)\right\} = b\in.$$

So, we have

$$\lim_{k\to\infty} M(r_{m(k)}, r_{n(k)}) \leq b\in. \qquad (3.6)$$

Now, letting $r = r_{m(k)}$ and $s = r_{n(k)}$ in (3.1), we obtain

$$\theta[b^3 d(r_{m(k)+1}, r_{n(k)+1})] \leq \phi[\theta(M(r_{m(k)}, r_{n(k)}))].$$

Letting $k \to \infty$ in the above inequality, applying the continuity of θ and using (3.6), we obtain

$$\theta\left(\frac{\in}{b^2} b^3\right) = \theta(b\in) \leq \theta\left[b^3 \lim_{k\to\infty} d(r_{m(k)+1}, r_{n(k)+1})\right]$$

$$\leq \phi\left[\theta\left(\lim_{k\to\infty} M(r_{m(k)}, r_{n(k)})\right)\right]$$

Therefore,

$$\theta(b\in) \leq \phi[\theta(b\in)] < \theta(b\in).$$

Since θ is increasing, we get $b\in < b\in$, which is a contradiction. Thus

$$\lim_{n,m\to} d(r_{m(k)}, r_{n(k)}) = 0.$$

Hence $\{r_n\}$ is a Cauchy sequence in S. By completeness of (S, d), there exists $w \in S$ such that

$$\lim_{n\to\infty} d(r_n, w) = 0.$$

Now, we show that $d(Tw, w) = 0$. Assume that $d(Tw, w) > 0$. Since $r_n \to w$ as $n \to \infty$, from Lemma 2.3, we conclude that

$$\frac{1}{b^2} d(w, Tw) \leq \limsup_{n\to\infty} d(Tr_n, Tw) \leq b^2 d(w, Tw).$$

Now, letting $r = r_n$ and $s = w$ in (3.3), we have

$$\theta(b^3 d(Tr_n, Tw)) \leq [\theta(M(r_n, w))], \forall n \in N,$$

Where

$$M(r_n, w) = a\max\{d(r_n, w), d(r_n, Tr_n), d(w, Tw), \frac{1}{b^2}(d(w, Tr_n) + d(r_n, Tw))\} + (1-a)d(r_n, w).$$

Taking the limit as $n \to \infty$, we have

$$\limsup_{n\to\infty} M(r_n, w) = \limsup_{n\to\infty} \max\{d(r_n, w), d(r_n, Tr_n), d(w, Tw), \frac{1}{b^2}(d(w, Tr_n) + d(r_n, Tw))\} = d(w, Tw).$$

Therefore,

$$\theta(b^3 d(Tr_n, Tw)) \leq \phi\left[\theta\left(a\max\{d(r_n, w), d(r_n, Tr_n), d(w, Tw), \frac{1}{2b^2}(d(w, Tr_n) + d(r_n, Tw) + (1-a)d(r_n, w))\}\right)\right]. \qquad (3.7)$$

Taking $n \to \infty$ in (3.7) and using the properties of ϕ and θ, we obtain

$$\theta\left(b^3 \frac{1}{b^2} d(w, Tw)\right) = \theta(bd(w, Tw))$$

$$\leq \theta\left[b^3 \lim_{n\to\infty} d(Tr_n, Tw)\right]$$

$$\leq \phi[\theta(d(w, Tw))] < \theta(d(w, Tw)).$$

By (θ_1), we get

$$bd(w, Tw) < d(w, Tw).$$

This implies that

$$(b-1)d(w, Tw) < 0$$

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which shows $b<1$, a contradiction. Hence $Tw=w$. Now, suppose that $u, w \in S$ are two fixed points of T such that $u \neq w$. Then we have

$$d(w,u)=d(Tw,Tu)>0.$$

Letting $r=w$ and $y=u$ in (3.1), we have

$$\theta(d(w,u))=\theta(d(Tu,Tw))\le \theta(b^3 d(Tu,Tw)) \le \phi[\theta(M(w,u))],$$

Where

$$M(w,u)=a\max\{d(w,u), d(w,Tw), d(u,Tu), \frac{1}{b^2}(d(u,Tw)+d(w,Tu))\}+(1-a)d(w,u) = d(w,u).$$

Therefore, we have

$$\theta(d(w,u))\le \phi[\theta(d(w,u))]<\theta(d(w,u)),$$

which implies that

$$d(w,u)<d(w,u),$$

a contradiction. Therefore $u=w$.

Corollary 3.1. In particular if $a=1$, we get Theorem 3.5 of Rossafi et al.[15].

Example 2. Let $S=A\cup B$, where $A=\{p^{n-1} \mid n\in N$ and $p\in\left(0,\frac{1}{5}\right)\}$ and $B=\{0\}$. Define $d: S\times S\to[0,\infty)$ by $d(r,s)=|r-s|^2$. Then (S,d) is a b–metric space with coefficient $b=2$. Define a mapping $T:S\to S$ by

$$T(r)=\begin{cases} r^n, \text{ if } r\in A \\ 1, \text{ if } r=0. \end{cases}$$

Then $T(r)\in S$. Let $\theta(t)=\sqrt{t}+1, \phi(t)=\frac{t+1}{2}$. It is obvious that $\theta\in\Theta$ and $\phi\in\Phi$ is Gregus type $\theta-\phi$ contraction. Consider the following possibilities:

Case: 1 Let $r=p^{n-1}, s=p^{m-1}$ for $m>n\ge 1$. Then

$$d(Tr,Ts)=(p^{n(n-1)}-p^{n(m-1)})^2.$$

So

$$\theta[b^3 d(Tr,Ts)]=\sqrt{8}(p^{n(n-1)}-p^{n(m-1)})+1$$

and

$$\phi[\theta(d(r,s))]=\phi[\theta(p^{n-1}-p^{m-1})^2] = \frac{(p^{n(n-1)}-p^{n(m-1)})}{2p}+1.$$

On the other hand

$$\theta[b^3 d(Tr,Ts)]-\phi[\theta(d(r,s))]=\sqrt{8}(p^{n(n-1)}-p^{n(m-1)}) +1-\left[\frac{(p^{n(n-1)}-p^{n(m-1)})}{2p}+1\right] = \left(\sqrt{8}-\frac{1}{2p}\right)(p^{n(n-1)}-p^{n(m-1)}) \le 0.$$

Thus

$$\theta[b^3 d(Tr,Ts)]\le\phi[\theta(d(r,s))] \le \phi[\theta(a\max\{d(r,s), d(r,Tr), d(s,Ts),. \frac{d(r,Ts)+d(Tr,s)}{2b^2}\}+(1-a)d(r,s))]$$

Case: 2 Let $r=p^{n-1}, s=0$. Then $T(r)=p^{n(n-1)}, T(s)=0$, then $d(Tr,Ts)=(p^n)^2$. So we have $\theta[b^3 d(Tr,Ts)]=\sqrt{8}p^{n(n-1)}+1$. Thus

$$M(r,s)=\phi[\theta(a\max\{d(r,s), d(r,Tr), d(s,Ts), \frac{d(r,Ts)+d(Tr,s)}{2b^2}\}+(1-a)d(r,s))] \ge d(r,s)=(p^{n-1})^2$$

and $\phi[\theta(d(r,s))]=\frac{p^{n-1}}{2}+1$

On the other hand

$$\theta[b^3 d(Tr,Ts)]-\phi[\theta(d(r,s))] =\sqrt{8}p^n+1-\left(\frac{p^{n-1}}{2}+1\right) =\left(\sqrt{8}-\frac{1}{2p}\right)p^n \le 0.$$

Then

$$\theta[b^3 d(Tr,Ts)]\le\phi[\theta(d(r,s))] \le \phi[\theta(a\max\{d(r,s), d(r,Tr), d(s,Ts), \frac{d(r,Ts)+d(Tr,s)}{2b^2}\}+(1-a)d(r,s))].$$

As a result, condition (3.1) is achieved. Hence T has an unique fixed point at $z=1$.

The following Figure(1) is graph of $T(r)$ for different values of $p\in\left(0,\frac{1}{5}\right)$. We see that in

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these graph unique fixed point at $r=1$.

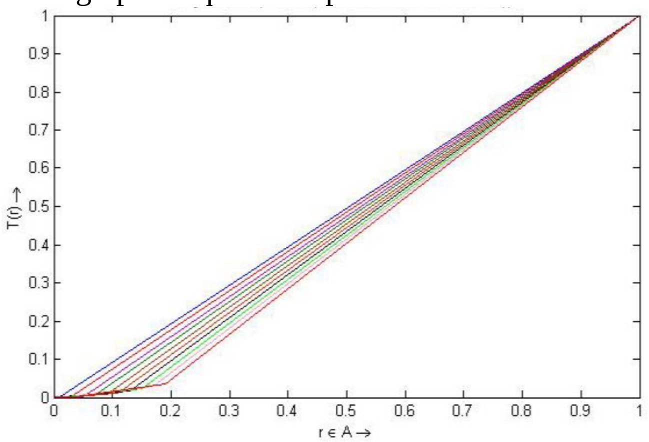

Figure 1 Graphical representation of the function $T(r)$

4. Application

In this section, as an application of Theorem 3.2, we present the following result which provides a unique solution to nonlinear integral equations of Fredholm type:

$$r(y)=\lambda\int_p^q K(y,x,r(x))\,dx, \qquad (4.1)$$

Where $p,q\in R, r\in C([p,q],R)$ and a continuous function $K:[p,q]^2\times R\to R$ is given.

Theorem 4.1. Assume that the kernel function K satisfies the condition:

$$|K(y,x,r(x))-K(y,x,s(x))|$$
$$\leq \frac{1}{b^2}(|r(x)-s(x)|)\forall x,y\in[p,q] \text{ and } r,s\in R$$

and consider the nonlinear integral equation problem (4.1). Then for some constant λ based on the constant b, the equation (4.1) has a unique solution $r\in C([p,q])$.

Proof. Let $T:X\to X$ be defined by

$Tr(y)=\lambda\int_p^q K(y,x,r(x))dx, \forall r\in X$, where $X=C([p,q])$. Let $d:X\times X\to[0,\infty)$ be given by

$$d(r,s)=(\max_{y\in[p,q]}|r(y)|-|s(y)|)^2\forall r,s\in X.$$

It is evident that (X,d) forms a complete b-metric space. To solve the integral equation (4.1), we need to find the condition on λ under which the operator has a unique fixed point. Assume that $r,s\in X$ and $x,y\in[p,q]$. Then we have

$$|Tr(y)-Ts(y)|^2=|\lambda|^2\,(|\int_p^q K(y,x,r(x))\,dx - \int_p^q K(y,x,s(x))\,dx|)^2$$
$$=|\lambda^2\,||\int_p^q (K(y,x,r(x))-K(y,x,s(x)))\,dx|^2$$
$$\leq|\lambda^2\,||\int_p^q (K(y,x,r(x))-K(y,x,s(x)))\,dx|^2$$
$$\leq|\lambda^2|\,|\int_p^q(\frac{1}{b^2}(|r(x)-s(x)|)\,dx)^2$$
$$=|\lambda^2|\,\frac{1}{b^4}[\int_p^q(|r(x)|-|s(x)|)\,dx]^2$$
$$=|\lambda^2|\,\frac{1}{b^4}[\int_p^q(|r(x)|-|s(x)|)\,dx]^2.$$

This shows that

$$\max_{y\in[p,q]}(|Tr(y)-Ts(y)|^2)$$
$$=\max_{y\in[p,q]}|\lambda|^2\int_p^q|(K(y,x,r(x))-K(y,x,s(x)))\,dx]^2$$
$$\leq\max_{y\in[p,q]}|\lambda|^2\int_p^q(\frac{1}{b^2}(|r(x)-s(x)|)\,dx)^2$$
$$\leq|\lambda|^2\frac{1}{b^4}\int_p^q((\max_{y\in[p,q]}(|r(y)-s(y)|)\,dr)^2.$$

Since, $d(Tr,Ts)>0$ and $d(r,s)>0\forall r\neq s$, we can take natural Log both sides and obtain

$$\log[b^3d(Tr,Ts)]$$
$$=\log[b^3|\lambda|^2\max_{y\in[p,q]}\int_p^q|(K(y,x,r(x))-K(y,x,s(x)))\,dx|^2]$$
$$\leq\log[(\frac{|\lambda|}{b})^2\int_p^q(\max_{y\in[p,q]}(|r(y)-s(y)|)\,dr)^2]$$
$$=[\log\{\int_p^q(\max_{y\in[p,q]}(|r(y)-s(y)|)\,dr)^2\}]^{(\frac{|\lambda|}{b})^2},$$

assuming that $|\lambda|<b$, this shows that

$$\log[b^3d(Tr,Ts)]$$
$$\leq[\log\{\int_p^q(\max_{y\in[p,q]}(|r(y)-s(y)|)\,dr)^2\}]^{\alpha}.$$

Hence

$$F(b^3d(Tr,Ts))+\phi(d(r,s))\leq F(d(r,s)),$$

$\forall r,s\in X$ with $\theta(x)=\log(x), \phi(x)=x^{\alpha}$ and $\alpha=(\frac{|\lambda|}{b})^2$. Thus T holds the condition (3.1). Therefore, the nonlinear Fredholm integral (4.1) has a unique solution.

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